ISRAEL JOURNAL OF MATHEMATICS 95 (1996), 253–280

RATIONAL CENTRAL SIMPLE ALGEBRAS

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Dedicated to the memory of E. C. Posner

ABSTRACT

We introduce a notion of rationality (called toroidal or t-rationality) for central simple algebras which extends Demazure's characterization of rational algebraic varieties via torus actions. We prove a structure theorem for t-rational central simple algebras and study the interplay among t-rationality, crossed products and rationality of the center in the setting of universal division algebras.

* Partially supported by an NSF postdoctoral fellowship.

** Partially supported by NSF grant DMS 9201465 and by FAPEAL (Alagoas, Brazil).

Received April 7, 1994

1. Introduction

Throughout this paper k will be a fixed algebraically closed base field, all rings will be PI-algebras over k and all homomorphisms between them will be k-algebra homomorphisms. All central simple algebras will be finite-dimensional over their centers which, in turn, are always assumed to be finitely generated field extensions of k. We will usually denote central simple algebras by A, division algebras by D, and their centers by K. The universal division algebra of m generic $n \times n$-matrices over k will be denoted by $D_{m,n}$ and its center by $K_{m,n}$.

Recall that a field extension $k \subset L$ is called **rational** if $L = k(x_1, \ldots, x_d)$ for some algebraically independent elements $x_1, \ldots, x_d$. In this paper we extend this notion of rationality to finite-dimensional central simple algebras. We call it **toroidal rationality** or **t-rationality** for short. A central simple algebra is defined to be **t-rational** if it admits a faithful action of a torus of maximal possible dimension; see Section 5 for details. This definition was motivated by Demazure's characterization of rational algebraic varieties via torus actions [De, p. 521] and our own investigation of torus actions on non-commutative rings [RV$_1$]. Demazure's result implies, in particular, that in the case of fields our definition of rationality coincides with the usual one.

We are especially interested in the following question: which universal division algebras $D_{m,n}$ are t-rational? (See Section 3 for background material on universal division algebras.) This question is related to two longstanding open problems: (a) Which universal division algebras $D_{m,n}$ are crossed products? and (b) Which universal division algebras have rational centers (over k)? A connection between these two problems was suggested by le Bruyn [B]. We show, in particular, that if $D_{m,n}$ is t-rational then it is a crossed product and has a rational center.

Our main result is the following classification of t-rational central simple algebras. Part (a) is proved in Section 6, and part (b) at the end of Section 7.

THEOREM 1.1:

(a) *A k-division algebra D with center K is t-rational if and only if D is isomorphic to a tensor product of symbol algebras*

$$(a_1, a_2, K, \omega_1) \otimes_K \cdots \otimes_K (a_{2r-1}, a_{2r}, K, \omega_r)$$

where $K = k(a_1, \ldots, a_d)$ is a purely transcendental extension of k of *transcendence degree* $d \geq 2r$, $\omega_i \in k$ *is a primitive* m_i*-th root of unity, and* $m_r \mid m_{r-1} \mid \cdots \mid m_1$.

(b) *A central simple algebra* A *is t-rational if and only if* $A \simeq M_n(D)$ *where* D *is a t-rational division algebra.*

The basic product of symbols in part (a) is isomorphic to the division algebra of fractions Q^d_Ω of the skew polynomial ring P^d_Ω; see Lemma 2.3. In this sense Theorem 1.1 is a non-commutative analogue of [De, Cor. 2, p. 521].

We also note that Theorem 1.1(a) resembles Tignol's decomposition theorem [T, 1.10] (and its generalizations by Draxl [Dr] and by Tignol and Wadsworth [TW]). In fact, our proof uses arguments similar to those in [T]; for details see Remark 6.3.

Applying Theorem 1.1 to universal division algebras we derive the following result which is proved in Section 8. Here we call an algebra D stably t-rational if it becomes t-rational after adjoining a finite number of central indeterminates.

THEOREM 1.2:

(a) *If the universal division algebra* $D_{m,n}$ *is stably t-rational then* n *is square-free and* $\mathrm{char}\,(k) \nmid n$.

(b) *Let* n *be a square-free integer. If the universal division algebra* $D_{m,p}$ *is stably t-rational for each prime divisor* p *of* n, *then* $D_{m,n}$ *is stably t-rational.*

The rest of this paper is organized as follows. Section 2 contains definitions and notational conventions which are used in the sequel. A brief summary of the background material on universal division algebras is presented in Section 3. In Section 4 we prove a result about skew-symmetric bilinear forms on free abelian groups. In Section 5 we introduce the notion of t-rationality for central simple algebras. Theorem 1.1(a) is proved in Section 6. In Section 7 we study torus actions on central simple algebras and prove Theorem 1.1(b). In Section 8 we derive several corollaries of Theorem 1.1 and prove Theorem 1.2. Our proof of part (b) relies on the remarkable recent paper of Saltman [Sa]. In Section 9 we construct a finite field extension F of $K_{2,n}$ of degree $(n-1)!$ such that $D_{2,n} \otimes F$ is a t-rational division algebra. This construction implies, in particular, that $D_{m,n}$ is t-unirational (i.e., is contained in a t-rational division algebra of degree n) for any $m, n \geq 2$ and that $D_{m,2}$ is t-rational for every $m \geq 2$.

We do not know whether or not the universal division algebra $D_{m,n}$ is t-rational for square-free integers $n \geq 3$. A positive answer to this question would, in particular, imply that $D_{m,n}$ is cyclic and has a rational center.

ACKNOWLEDGEMENT: The authors would like to thank B. Fein, D. Saltman and M. Schacher for useful discussions, and the referee for helpful comments.

Part of this work was done while the second author enjoyed the hospitality of the Departamento de Matemática Básica of the Universidade Federal de Alagoas (UFAL) in Maceió, Alagoas, Brazil. During this period he was supported by a fellowship from the Fundação de Amparo à Pesquisa do Estado de Alagoas, which in addition partially financed a shorter visit to UFAL by the first author. We also thank UFAL's Departamento de Matemática Aplicada for providing access to electronic mail, which greatly facilitated this collaborative research.

2. Preliminaries

We shall use the following notation.

m, n	integers ≥ 2,
k	algebraically closed base field,
$T = T_d = (k^*)^d$	d-dimensional torus,
$X_*(T)$	the group of characters of T,
ω	usually denotes a root of unity in k,
A	central simple algebra, usually with center K,
D	division algebra, usually with center K,
$Z(R)$	center of the ring R
(a, b, K, ω)	symbol algebra; See Definition 2.1(a),
P_Ω^d	skew polynomial ring; see Definition 2.2,
Q_Ω^d	division algebra of fractions of P_Ω^d,
$T_{m,n}(F)$	trace ring of m generic $n \times n$-matrices over F,
$C_{m,n}(F)$	center of $T_{m,n}(F)$,
$D_{m,n}(F)$	universal division algebra of m generic $n \times n$-matrices,
$K_{m,n}(F)$	center of $D_{m,n}(F)$.

We will usually write $G_{m,n}$ for $G_{m,n}(k)$, $D_{m,n}$ for $D_{m,n}(k)$, etc.

Definition 2.1: (a) Let K be a field, let $\omega \in k$ be a primitive m-th root of unity and let $a, b \in K$. The **symbol algebra** (a, b, K, ω) is the K-algebra given by generators x and y and relations $x^m = a$, $y^m = b$ and $xy = \omega yx$.

(b) We shall call a division algebra D a **basic product of symbols** if its center $K = k(a_1, \ldots, a_d)$ is a purely transcendental extension of k of transcendence degree d and

$$D \simeq (a_1, a_2, K, \omega_1) \otimes_K \cdots \otimes_K (a_{2r-1}, a_{2r}, K, \omega_r) \tag{1}$$

where $2r \leq d$, $\omega_i \in k$ is a primitive m_i-th root of unity and $m_r \mid m_{r-1} \mid \cdots \mid m_1$.

Definition 2.2: Let $\Omega = (\omega_1, \ldots, \omega_r)$ be an r-tuple of roots of unity in k where ω_i is a primitive m_i-th root of unity and $m_r \mid m_{r-1} \mid \cdots \mid m_1$. Let $d \geq 2r$. We define P^d_Ω to be the k-algebra given by generators $z_1, \ldots, z_d$ and relations $z_j z_l = c_{jl} z_l z_j$ with $c_{2i-1,2i} = \omega_i$ for $i = 1, \ldots r$ and $c_{jl} = 1$ for all other values of $j \leq l$. It is a skew polynomial ring. We denote the division algebra of fractions of P^d_Ω by Q^d_Ω.

LEMMA 2.3: *The algebra P^d_Ω is a PI-domain of degree $n = m_1 m_2 \cdots m_r$. Its division algebra of fractions Q^d_Ω is isomorphic to the basic product of symbols* (1).

Proof: The first statement is proved in [R_1, 3.3.6]. To prove the second assertion denote the generators of the i-th symbol algebra by x_{2i-1} and x_{2i}. That is, for $i = 1, \ldots, r$ we have $x_{2i-1}^{m_i} = a_{2i-1}$, $x_{2i}^{m_i} = a_{2i}$, and $x_{2i-1}x_{2i} = \omega_i x_{2i} x_{2i-1}$. Let ϕ: $P^d_\Omega \longrightarrow D$ be the map given by $\phi(z_i) = x_i$ for $i = 1, \ldots, r$ and $\phi(z_j) = a_j$ for $j = r+1, \ldots, d$. By [R_1, 3.3.6], ϕ maps the center of P^d_Ω isomorphically onto $k[a_1, \ldots, a_d]$. Thus ϕ is injective by [R_1, 1.6.27] and extends to an injective homomorphism ψ: $Q^d_\Omega \longrightarrow D$. The image of this homomorphism contains all x_i and all a_j. Thus ψ is an isomorphism. ∎

Note that Lemma 2.3 shows, in particular, that a basic product of symbols is indeed a division algebra. We note the following properties of the algebras Q^d_Ω.

THEOREM 2.4: *Let $D = Q^d_\Omega$ be as in Definition 2.2. Denote the center of D by K, and set $n = m_1 \cdots m_r$. Then*

(a) *If* char $(k) = p > 0$ *then* $p \nmid n$.

(b) *D is a crossed product for the group $\mathbb{Z}/m_1\mathbb{Z} \times \cdots \times \mathbb{Z}/m_r\mathbb{Z}$.*

(c) *If L is a subfield of D containing K then the Galois group of L over K is abelian.*

(d) *The exponent of D is equal to m_1.*

(e) *If n is square-free then $r = 1$ and D is cyclic.*

Proof: As seen above, n is the degree of D. (a) If char $(k) = p > 0$ then no m_i is divisible by p. Thus $n = m_1 \cdots m_r$ is not divisible by p. (b) and (c) follow from

[R_1, 3.3.6 and 3.3.9]. (d) is a consequence of Lemma 2.3, and (e) follows from (b). ∎

3. Universal division algebras

In this section we briefly recall a number of definitions and results concerning universal division algebras which will be used in the sequel.

Let F be a field and let $x_{ij}^{(l)}$ be mn^2 independent commuting variables; here $i, j = 1, \ldots, n$ and $l = 1, \ldots, m$. The algebra of generic matrices $G_{m,n}(F)$ is the F-subalgebra of the matrix algebra $M_n(F[x_{ij}^{(l)}])$ generated by the m generic matrices $X_1 = (x_{ij}^{(1)}), \ldots, X_m = (x_{ij}^{(m)})$. This is a domain of PI-degree n. Its division algebra of fractions is called the **universal division algebra** of m generic $n \times n$-matrices. We shall denote this division algebra by $D_{m,n}(F)$. The center of $D_{m,n}(F)$ is called the field of rational matrix invariants. It is generated (as a field extension of F) by the coefficients of the characteristic polynomials of the elements of $G_{m,n}$. We shall denote this field by $K_{m,n}(F)$. For details of this construction see [C, Section 12.6] or [F_3].

Throughout most of this paper we shall be interested in universal division algebras over $F = k$. In this case we will write $D_{m,n}$ and $K_{m,n}$ for $D_{m,n}(k)$ and $K_{m,n}(k)$, respectively. It is easy to see that if F is an extension of k then $D_{m,n}(F) = D_{m,n} \otimes_k F$; see, e.g., the proof of [Sa, Lemma 12].

THEOREM 3.1 (Procesi): *For every $m \geq 2$ the universal division algebra $D_{m+1,n}$ is isomorphic $D_{m,n}$ extended by n^2 central indeterminates. Specifically, if one embeds $D_{m,n}$ in $D_{m+1,n}$ in the natural way, then $D_{m+1,n} = D_{m,n}(c_{ij})$ where $i, j = 0, \ldots, n-1$, and the n^2 central elements $c_{ij} = \operatorname{tr}(X_1^i X_2^j X_{m+1})$ are algebraically independent over $K_{m,n}$.*

Proof: By [Pr, p. 255], the n^2 elements $c_{ij} \in K_{m+1,n}$ are algebraically independent over $K_{m,n}$ and $K_{m+1,n} = K_{m,n}(c_{ij})$; see also [$R_1$, 3.3.31]. Thus the natural inclusion of $D_{m,n}$ into $D_{m+1,n}$ extends to an isomorphism. ∎

Note that we can also write $D_{m+1,n}$ as $D_{m,n}(d_{ij})$ where the d_{ij} are elements of $K_{m+1,n}$ satisfying

$$X_{m+1} = \sum_{i,j=0}^{n-1} d_{ij} X_1^i X_2^j .$$

Recall that division algebras D and E are **stably isomorphic** if the division algebras $D(x_1, \ldots, x_r)$ and $E(y_1, \ldots, y_s)$ are isomorphic for some central indeterminates $x_1, \ldots, x_r, y_1, \ldots, y_s$. In particular, Theorem 3.1 implies that $D_{m_1,n}$ is stably isomorphic to $D_{m_2,n}$ for any $m_1, m_2 \geq 1$.

THEOREM 3.2 (Saltman): *Let $n = ab$ where a and b are relatively prime positive integers. Then the universal division algebra $D_{m,n}$ is stably isomorphic to the algebra $D_{m,a} \otimes_F D_{m,b}$, where F is the field compositum of $K_{m,a}$ and $K_{m,b}$ over k.*

Proof: This is implicitly contained in the proof of [Sa, Theorem 13]. For completeness we supply a short explanation. Write

$$D_{m,n} = D(a) \otimes_{K_{m,n}} D(b) \tag{2}$$

where $D(a)$ and $D(b)$ are division algebras of degree a and b respectively. Let K be the field compositum of $K_{m,n}$ and F over k. Define L_a to be the function field of the Brauer–Severi variety given by $D_{m,a} \otimes_K D(a)^\circ$ over K; similarly for L_b. Let L' be the field compositum of L_a and L_b over K. In particular, $D_{m,a} \otimes_{K_{m,a}} L' = D(a) \otimes_{K_{m,n}} L'$ and $D_{m,b} \otimes_{K_{m,b}} L' = D(b) \otimes_{K_{m,n}} L'$. Combining this with (2) we obtain

$$\begin{aligned} D_{m,n} \otimes_{K_{m,n}} L' &= (D_{m,a} \otimes_{K_{m,a}} L') \otimes_{L'} (D_{m,b} \otimes_{K_{m,b}} L') \\ &= (D_{m,a} \otimes_F D_{m,b}) \otimes_F L'. \end{aligned}$$

The proof of [Sa, Theorem 13] shows that L' is rational over $K_{m,n}$ and stably isomorphic to F. ∎

4. Skew-symmetric forms

It is well-known that a skew-symmetric bilinear form ψ on a d-dimensional real vector space can always be written in the form

$$\psi = g_1^* \wedge g_2^* + \cdots + g_{2r-1}^* \wedge g_{2r}^*$$

for some basis $\{g_1, \ldots, g_d\}$ and some $r \leq d/2$; see, e.g., [St, I.5.1]. Here $\{g_1^*, \ldots, g_d^*\}$ is the dual basis of V^*. A similar statement is true for skew-symmetric forms on finite abelian groups; see [dR, Sect. 19] and [W, Sect. 4]. In this section we prove the following variant of these results.

PROPOSITION 4.1: *Let V be a free $\mathbb{Z}$-module of rank d, let m be a positive integer and let $\psi\colon V \times V \longrightarrow \mathbb{Z}/m\mathbb{Z}$ be a skew-symmetric bilinear form. Then there exists a $\mathbb{Z}$-module basis $g_1, \ldots, g_d$ of V such that*

$$\psi = c_1 g_1^* \wedge g_2^* + \cdots + c_r g_{2r-1}^* \wedge g_{2r}^*.$$

Here $2r \leq d$, $c_1, \ldots, c_r \in \mathbb{Z}/m\mathbb{Z}$ and $\operatorname{ord}(c_r) \mid \operatorname{ord}(c_{r-1}) \mid \cdots \mid \operatorname{ord}(c_1)$.

Proof: We begin with the following well-known lemma; see, e.g., [Lam, I.5.5 and I.4.6]. For completeness we supply a short direct proof below.

LEMMA 4.2: *Suppose $a_1, \ldots, a_d$ are integers and $\gcd(a_1, \ldots, a_d) = 1$. Then there exists a $d \times d$ integral matrix M of determinant ± 1 whose first row is $(a_1, \ldots, a_d)$.*

Proof: We say that the d-tuple of integers $a = (a_1, \ldots, a_d)$ is completable if there exists a $d \times d$ integral matrix M of determinant ± 1 whose first row is $(a_1, \ldots, a_d)$. Note that for any integral matrix N of determinant ± 1, aN is completable if and only if a is. In particular, we can permute the a_i -s, replace a_2 by $a_2 + na_1$ where $n \in \mathbb{Z}$, or multiply a_i by -1. If we obtain a completable d-tuple as a result of these operations then the original d-tuple was completable. Since the above-described operations allow us to perform the Euclidean algorithm, we can use them to construct the d-tuple $(1, 0, \ldots, 0)$. Since $(1, 0, \ldots, 0)$ is obviously completable, our original d-tuple must be completable as well. ∎

LEMMA 4.3: *Let $r = \max\{\operatorname{ord}\psi(e, f)\colon e, f \in V\}$. Then there is a basis $\{e_1, \ldots, e_d\}$ of V such that $\psi(e_1, e_2)$ is of order r.*

Proof: Suppose $\operatorname{ord}\psi(e, f) = r$ for some $e, f \in V$. We may assume $e \notin nV$ for any integer $n \geq 2$; otherwise replace e by e/n. Then Lemma 4.2 says that there is a basis $\{f_1, \ldots, f_d\}$ of V with $f_1 = e$. Suppose

$$f = a_1 f_1 + \cdots + a_d f_d$$

for some $a_1, \ldots, a_d \in \mathbb{Z}$. For $i = 2, \ldots, d$ let

$$b_i = a_i / \gcd(a_2, \ldots, a_d).$$

Since $\psi(e, b_2 f_2 + \ldots + b_d f_d)$ divides $\psi(e, f)$ in $\mathbb{Z}/m\mathbb{Z}$, its order must also be r. Hence, we may assume without loss of generality that

$$f = b_2 f_2 + \cdots + b_d f_d\ .$$

By Lemma 4.2 there exists a basis $\{e_2, \dots, e_d\}$ of $\mathrm{Span}(f_2, \dots, f_d)$ such that $f = e_2$. Set $e_1 = f_1 = e$. We have thus constructed a basis $\{e_1, \dots, e_d\}$ of V with the property that $\psi(e_1, e_2)$ has order r. ∎

We can now finish the proof of Proposition 4.1 by induction on d. For $d \leq 2$ the proposition is obvious. Suppose it holds for $d-2$. Choose a basis $\{e_1, \dots, e_d\}$ of V as in Lemma 4.3. Note that $\psi(e_1, e_2)$ is a generator of the subgroup $\psi(e_1, V)$ of $\mathbb{Z}/m\mathbb{Z}$. Indeed, this subgroup is cyclic, and the order of its generator cannot be strictly greater then the order of $\psi(e_1, e_2)$. Similarly $\psi(e_1, e_2)$ is a generator of $\psi(e_2, V)$. Therefore, for every $i = 3, \dots, d$ we can find $a_i, b_i \in \mathbb{Z}$ such that

$$\psi(e_1, e_i + a_i e_1 + b_i e_2) = \psi(e_1, e_i) + b_i \psi(e_1, e_2) = 0$$

and

$$\psi(e_2, e_i + a_i e_1 + b_i e_2) = \psi(e_2, e_i) - a_i \psi(e_1, e_2) = 0.$$

Now let $f_1 = e_1$, $f_2 = e_2$, and $f_i = e_i + a_i e_1 + b_i e_2$. Note that $\{f_1, \dots f_d\}$ is a new $\mathbb{Z}$-module basis for V. By our choice of a_i and b_i

$$\psi = \psi(f_1, f_2) f_1^* \wedge f_2^* + \psi_0$$

where

$$\psi_0 = \sum_{3 \leq i < j} \psi(f_i, f_j) f_i^* \wedge f_j^*.$$

Let $g_1 = f_1$ and $g_2 = f_2$. Applying our induction assumption to ψ_0, we can find a basis $\{g_3, \dots, g_d\}$ of $\mathrm{Span}(f_3, \dots, f_d)$ such that

$$\psi = c_1 g_1^* \wedge g_2^* + \cdots + c_r g_{2r-1}^* \wedge g_{2r}^*,$$

where

$$r = \mathrm{ord}(c_1) = \max\{\, \mathrm{ord}\psi(e, f) \colon e, f \in V\}$$

and

$$\mathrm{ord}(c_r) \mid \mathrm{ord}(c_{r-1}) \mid \cdots \mid \mathrm{ord}(c_2).$$

If $r = 1$, we are done. If $r \geq 2$ it remains to show that $\mathrm{ord}(c_2) \mid \mathrm{ord}(c_1)$. Note that $\psi(\mathbb{Z}g_1 + \mathbb{Z}g_3, g_2 + g_4) = \mathbb{Z}c_1 + \mathbb{Z}c_2$. By the maximality of $\mathrm{ord}(c_1)$, this implies that $\mathbb{Z}c_1 + \mathbb{Z}c_2 = \mathbb{Z}c_1$. Consequently, $\mathrm{ord}(c_2) \mid \mathrm{ord}(c_1)$, as claimed. ∎

5. Torus actions and t-rationality

By a torus of dimension (or rank) d we shall mean the algebraic group $T = (k^*)^d$. A torus action on a k-algebra R is a group homomorphism $\varphi: T \longrightarrow \mathrm{Aut}_k(R)$ where $\mathrm{Aut}_k(R)$ is the group of k-algebra automorphisms of R. The action is faithful if φ is injective. If $t \in T$ and $a \in R$ then we shall write $t(a)$ or ta for $\varphi(t)(a)$ when the reference to the action φ is clear from the context. The set of invariant elements of R, i.e., elements a such that $ta = a$ for every $t \in T$, will be denoted by R^T.

A non-zero element $a \in R$ is called **homogeneous** if there exists an algebraic character $\chi: T \longrightarrow k^*$ such that $t(a) = \chi(t)a$ for every $t \in T$. We say that χ is the character associated to a. The T-action on R is called **rational** if every element of R can be written as a sum of homogeneous elements. Note that a T-action on a commutative algebra R is rational if and only if the induced action of T on $\mathrm{Spec}(R)$ is given by a morphism of schemes $T \times \mathrm{Spec}(R) \longrightarrow \mathrm{Spec}(R)$.

LEMMA 5.1: *Given a rational torus action on a k-algebra R,*

(a) *every $a \in R$ can be uniquely written as a sum of homogeneous elements with distinct associated characters;*

(b) *the restricted action on the center $Z(R)$ of R is also rational.*

Proof: (a) Follows from linear independence of characters.

(b) Write a central element $a \in Z(R)$ as a sum $a_1 + \cdots + a_r$ where the elements a_i are homogeneous with distinct associated characters. It is enough to show that each a_i is central. In fact, we only need to check that a_i commutes with every homogeneous element $h \in R$. This follows from part (a) applied to the element $ah = ha \in R$. ∎

There are no non-trivial rational torus actions on a division algebra; see, e.g., [V, appendix]. We shall be interested in torus actions satisfying the following weaker condition.

Definition 5.2: Let A be a central simple algebra. A torus action on A is **algebraic** if every element of A can be written as bc^{-1} where b is a sum of homogeneous elements and $c \neq 0$ is a sum of central homogeneous elements.

Algebraic actions are related to rational actions in the following simple way.

LEMMA 5.3: *Let A be a central simple algebra, and let T be a torus acting k-linearly on A. Then the action of T on A is algebraic if and only if A contains* *a T-stable prime k-subalgebra R such that*

(a) *T acts rationally on R, and*

(b) *every element of A is of the form bc^{-1} where $b \in R$ and $0 \neq c \in Z(R)$.*

Proof: If the action of T on A is algebraic, take R to be the subalgebra of A generated by the homogeneous elements. Then T acts rationally on R and on the center $Z(R)$; see Lemma 5.1. Since A is a central localization of R, R is prime. This proves one direction. Conversely, suppose A contains a subalgebra R satisfying (a) and (b). Then T acts rationally on the center $Z(R)$ of R. By Posner's theorem*, A is a central localization of R. So every element of A is of the form $a = bc^{-1}$, where b and c are sums of homogeneous elements of R and $Z(R)$, respectively. ∎

LEMMA 5.4: *Suppose a torus T acts faithfully and algebraically on a division algebra D. Then every character of T occurs as an associated character for some homogeneous element of D.*

Proof: Since D is a division algebra, the characters associated to homogeneous elements of D form a subgroup V of the full character group $X_*(T)$. Recall that $X_*(T)$ is a free abelian group of rank $d = \dim(T)$. Thus V is a free abelian group of rank $r \leq d$. We want to prove $V = X_*(T)$.

Let $\chi_1, \ldots, \chi_r$ be a basis of V. Then our action $T \longrightarrow \mathrm{Aut}_k D$ factors through $\phi\colon T \longrightarrow S = (k^*)^r$ where $\phi(t) = (\chi_1(t), \ldots, \chi_r(t))$. Since the T-action on D is faithful, ϕ is injective. This implies $d = r$. Moreover, the factor group $S/\phi(T)$ is both irreducible and 0-dimensional. Hence, $S/\phi(T) = \{1\}$, i.e., ϕ is surjective. This proves that ϕ is an isomorphism and thus $X_*(T) = \phi^*(X_*(S)) = V$, as desired. ∎

We now proceed to define the notion of t-rationality. Our starting point is the following proposition; see [RV_1, 1.3 and 3.6].

PROPOSITION 5.5: *Let A be a central simple k-algebra of transcendence degree d and matrix size n. Assume that A admits a faithful algebraic action of a torus T. Then*

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* Posner's original theorem [Po] does not involve central localization. The stronger version we use here (which is also commonly referred to as Posner's theorem) was independently discovered by at least seven mathematicians; see [R_1, p. 53 and p. 340] and [F_3, p. 15].

(a) $\dim T \leq d + n - 1$.
(b) *If the action of T fixes the center of A pointwise, then* $\dim T \leq n - 1$.
(c) *If* $A = D$ *is a division algebra with center* K, *then* $\dim T = \mathrm{trdeg}(K/K^T)$.

Note that Proposition 5.5 may be viewed as an affine non-commutative generalization of [De, Cor 1, p. 521].

Definition 5.6: Let A be a central simple algebra which is finite-dimensional over its center K. Assume that the matrix size of A is n, and that K is an extension of k of transcendence degree d. Then A is called **toroidally rational** or **t-rational** (over k) if it admits a faithful algebraic action of a torus of dimension $d + n - 1$. Moreover, A is called **stably t-rational** if it becomes t-rational after adjoining a finite number of central indeterminates.

Remark 5.7: Note that our definition of t-rationality mimics the statement of [De, Cor 2b, p. 521] in the non-commutative setting. In fact, [De, Cor 2, p. 521] immediately implies that a field is t-rational if and only if it is a purely transcendental extension of k.

Example 5.8: Let P^d_Ω be the skew polynomial ring of Definition 2.2. Then the division algebra of fractions Q^d_Ω of P^d_Ω is t-rational. Indeed, the d-dimensional torus $T = (k^*)^d$ acts faithfully and rationally on P^d_Ω by $t(z_i) = t_i z_i$ for $t = (t_1, \ldots, t_d) \in T$. Hence, Q^d_Ω is t-rational by Lemma 5.3.

In the next section we will prove that every t-rational division algebra is, in fact, isomorphic to Q^d_Ω for some choice of Ω and d.

LEMMA 5.9: *If A is a t-rational central simple algebra then so is* $A(x)$ *for a central indeterminate* x.

Proof: If A is t-rational via a faithful algebraic action of a torus T, extend the T-action on A to an algebraic $T \times k^*$-action on $A(x)$ in the obvious way. ∎

PROPOSITION 5.10:

(a) *Let T be a torus acting algebraically on a central simple algebra A of degree n and let K be the center of A. Then the induced action of T on K is algebraic.*
(b) *Assume additionally that* $A = D$ *is a division algebra, and that the T-action on D is faithful. Then the subgroup S of T which acts trivially on K is finite.*

Proof: (a) Let R be the k-subalgebra of A generated by all homogeneous elements and let Z be the center of R. Then T acts rationally on R and Z; see Lemma 5.1. By our assumption A is the total ring of fractions of R. By Posner's theorem, the center K of A is equal to the field of fractions of Z. Since T acts rationally on Z, it acts algebraically on K.

(b) Suppose $t \in S$. Let h be a non-zero homogeneous element of D with character χ. Then $\det(h)$ is a homogeneous element of K. (Note that det commutes with every automorphism of D.) The associated character of this homogeneous element is χ^n where n is the degree of D. Consequently, $\chi(t^n) = \chi(t)^n = 1$. Since this is true for every character of T (see Lemma 5.4), we conclude that t^n acts trivially on D, so that $t^n = 1$. Hence, S is contained in the n-torsion subgroup of T, which is finite. ∎

6. Proof of Theorem 1.1(a)

In this section we prove Theorem 1.1(a). In view of Lemma 2.3 and Example 5.8, we can restate it as follows.

THEOREM 6.1: *Every t-rational division algebra D is isomorphic to the division algebra of fractions Q_Ω^d of P_Ω^d for some choice of $\Omega = (\omega_1, \dots, \omega_r)$ and some integer $d \geq 2r$.*

It is worth noting that as a consequence of this result, Theorem 2.4 applies to every t-rational division algebra.

Proof: We begin with the following lemma.

LEMMA 6.2: *Let D be a t-rational division algebra of degree n, and let T be a torus of dimension $d = \mathrm{trdeg}_k D$ acting faithfully and algebraically on D. Then*

(a) $D^T = k$.
(b) *Every homogeneous component of D has dimension 1 over k.*
(c) *Any two homogeneous elements x and y of D commute up to an n-th root of unity. That is, $xyx^{-1}y^{-1}$ is an n-th root of unity in k.*
(d) *The n-th power of every homogeneous element is central in D.*

Proof: (a) Denote the center of D by K. By Proposition 5.5(c), $\mathrm{trdeg}(K/K^T) = d$, i.e., K^T is an algebraic extension of k. Since k is algebraically closed, this means $K^T = k$. Assume $x \in D^T$. Then the coefficients of the characteristic polynomial of x are also fixed by T, i.e. they lie in $K^T = k$. Hence, x satisfies a polynomial over k. Since k is algebraically closed, this implies $x \in k$.

(b) If x and y are homogeneous elements with the same associated character, then $yx^{-1} \in D^T = k$, that is, $y \in kx$.

(c) We have $xyx^{-1}y^{-1} \in D^T = k$. Since this element has determinant 1, it has to be an n-th root of unity.

(d) follows easily from (c). ∎

We are now ready to finish the proof of Theorem 6.1. Let T be a torus of dimension d acting faithfully and algebraically on D. We define a skew-symmetric bilinear form ψ on the free abelian group $V = X_*(T)$ as follows. Recall that for every character $\chi \in V$ there exists a non-zero homogeneous element $x_\chi \in D$ with associated character χ; see Lemma 5.4. Let

$$\psi(\chi, \eta) = x_\chi x_\eta (x_\chi)^{-1} (x_\eta)^{-1}.$$

By Lemma 6.2(b), $\psi(\chi, \eta)$ depends only on χ and η and not on the particular choice of x_χ and x_η. By Lemma 6.2(c) this form takes values in the (cyclic) group of the n-th roots of unity in k. Note that we are writing this group multiplicatively. By Proposition 4.1 we can choose a basis $\chi_1, \dots, \chi_d$ of V such that $\psi(\chi_{2i-1}, \chi_{2i}) = \omega_i$ is an n-th root of unity for $i = 1, \dots, r \le d/2$ and $\psi(\chi_j, \chi_l) = 1$ for all other $j \le l$. Moreover, we may assume $\operatorname{ord}(\omega_{i+1}) \mid \operatorname{ord}(\omega_i)$ for $i = 1, \dots, r-1$.

Now let $x_i = x_{\chi_i} \in D$ be a homogeneous element with associated character χ_i for $i = 1, \dots, d$ and let R be the k-subalgebra of D generated by these elements. Note that $x_j x_l = \psi(\chi_j, \chi_l) x_l x_j$.

We claim that D is the division algebra of fractions $\mathrm{Q}(R)$ of R. It suffices to prove that every homogeneous element y of D belongs to $\mathrm{Q}(R)$. Let χ be the associated character of y. Then $\chi = \chi_1^{a_1} \cdots \chi_r^{a_r}$ for some integers a_i. Consequently, y and $x_1^{a_1} \cdots x_r^{a_r}$ belong to the same homogeneous component. By Lemma 6.2(b), the homogeneous components are one-dimensional over k. Thus $y \in \mathrm{Q}(R)$, proving $\mathrm{Q}(R) = D$.

We will now prove that R is isomorphic to P_Ω^d. Indeed, let ϕ be the k-algebra homomorphism $P_\Omega^d \longrightarrow R$ which sends z_i to x_i for $i = 1, \dots, d$. By our choice of x_i, ϕ is well-defined and surjective. It is injective by linear independence of characters. This proves that R is isomorphic to P_Ω^d. Consequently, $D = \mathrm{Q}(R)$ is isomorphic to Q_Ω^d. ∎

Remark 6.3: If H is the group of homogeneous elements of D then $H/H \cap K$ is an armature (i.e., a finite abelian subgroup of D^*/K^*) of order n^2; see [T, 1.5]. Applying the results of [T] to this armature, one obtains a decomposition of D as a product of symbol algebras. However, the product in [T, 1.10] may not be basic. In order to write D as a basic product of symbols, we worked with the free abelian group $V = X_*(T)$ instead of the finite group $H/H \cap K$ and appealed to Proposition 4.1 instead of [T, 1.8].

7. Torus actions on central simple algebras

In this section we study algebraic torus actions on central simple algebras. Our main result is Theorem 7.5. At the end of this section we prove Theorem 1.1(b). We begin with the following lemma.

LEMMA 7.1: *Consider a t-rational action of a torus T on a k-algebra R. Suppose R contains a non-zero nilpotent element. Then R contains a non-zero homogeneous nilpotent element.*

Proof: Recall that the characters of T form a free $\mathbb{Z}$-module of rank $d = \dim(T)$, denoted by $X_*(T)$. We can identify $X_*(T)$ with $\mathbb{Z}^d$ by identifying the character χ with the tuple $(m_1, \ldots, m_d)$ if

$$\chi(s_1, \ldots, s_d) = s_1^{m_1} \cdots s_d^{m_d}.$$

The usual Euclidean norm on $\mathbb{Z}^d \subset \mathbb{R}^d$ induces thus a norm $N: X_*(T) \longrightarrow \mathbb{R}$ on the set of characters. In concrete terms, for χ as above,

$$N(\chi) = \sqrt{m_1^2 + \cdots + m_d^2}.$$

To simplify the use of the norm N, we will write the group $X_*(T)$ additively throughout this proof.

Suppose $x \in R$ is nilpotent. Write x as a sum of homogeneous elements:

$$x = x_1 + \cdots + x_r.$$

Let $\chi_i: T \longrightarrow k^*$ be the character of T associated to the homogeneous element x_i. We may assume that the characters χ_i are distinct, and that $N(\chi_1) \geq N(\chi_i)$ for all i. We claim that x_1 is nilpotent. Indeed, we know that

$$(x_1 + \cdots + x_r)^w = x^w = 0$$

for some positive integer w. Expanding this expression and collecting the terms with the same associated character, we conclude that

$$\sum_{\chi_{i_1}+\cdots+\chi_{i_w}=\mu} x_{i_1}\cdots x_{i_w} = 0 \tag{3}$$

for every character $\mu\colon T \longrightarrow k^*$. If we now set $\mu = w\chi_1$ then the above sum will only contain one term, namely x_1^w. Indeed, $\chi_{i_1} + \cdots + \chi_{i_w} = \mu$ implies

$$wN(\chi_1) = N(\mu) = N(\chi_{i_1} + \cdots + \chi_{i_w}) \leq N(\chi_{i_1}) + \cdots + N(\chi_{i_w}) \leq wN(\chi_1).$$

Since we are assuming that $N(\chi_1) \geq N(\chi_{i_j})$, this implies $N(\chi_{i_j}) = N(\chi_1)$ for all j. Hence, the equality $\chi_{i_1} + \cdots + \chi_{i_w} = w\chi_1$ implies $\chi_{i_1} = \cdots = \chi_{i_w} = \chi_1$. In other words, when $\mu = w\chi_1$, equality (3) becomes $x_1^w = 0$. Thus x_1 is a non-zero homogeneous nilpotent element of R. ∎

LEMMA 7.2: *Let T be a torus acting algebraically on a central simple algebra A. Then the nilradical N of A^T is nilpotent, and A^T/N is semisimple Artinian.*

We will later show that in fact $N = 0$, so that A^T is itself semisimple Artinian; see Corollary 8.6.

Proof: Denote by R the subalgebra of A generated by the homogeneous elements. Then $R^T = A^T$, and R is prime. By [LVV, II.3.4.2], R is an Azumaya algebra and thus finite over its center. By [V, 2.5], R^T contains a set S of elements which are central (and thus regular) in R such that the localization R^TS^{-1} is Artinian modulo its nilradical N, which is nilpotent. But all elements of R^T which are central in R (and thus A) are invertible in R^T. Thus $A^T = R^T = R^TS^{-1}$. ∎

PROPOSITION 7.3: *Suppose a torus T acts algebraically on a central simple algebra A. Assume that A is not a division algebra. Then the fixed algebra A^T contains a non-trivial idempotent.*

Proof: Assume the contrary: A^T has no non-trivial idempotents.

We first show that under these assumptions every element of A^T which is not nilpotent is invertible. By Lemma 7.2, the nilradical of A^T is nilpotent, and A^T/N is semisimple Artinian. Since idempotents can be lifted modulo nilpotent ideals, and since A^T has no non-trivial idempotents, the semisimple Artinian algebra A^T/N is a division algebra. Now let x be a non-nilpotent element of A^T. Then x is invertible modulo N, i.e., for some $y \in A^T$, $xy = 1 + n$ with n

nilpotent. But elements of the form $1+n$ are invertible. Thus x is invertible, as claimed.

Denote the center of A by K. Let I be the K-vector space spanned by the homogeneous zero-divisors in A. Since I is closed under multiplication by homogeneous elements, it is an ideal of A. Since we are assuming that A is simple, this implies that either $I = 0$ or $I = A$. We will show that neither can happen. This will contradict the assumption that A^T has no non-trivial idempotents.

First we rule out the case $I = 0$. Let R be the k-subalgebra of A generated by the homogeneous elements. Since A is not a division algebra, it contains a non-zero nilpotent element. After multiplying this element by a scalar, we may assume that it lies in R. Since T acts rationally on R, Lemma 7.1 says that R contains a non-zero homogeneous nilpotent element. Hence, $I \neq 0$.

It remains to rule out the case $I = A$. We shall do this by showing that I is nilpotent. First note that a homogeneous zero divisor $x \in A$ is necessarily nilpotent. Indeed, assume the contrary. Then by the Cayley–Hamilton Theorem $c_i(x) \neq 0$ for some coefficient $c_i(x)$ of the characteristic polynomial of x. The element $x^i/c_i(x)$ of A^T is not nilpotent, so it is invertible. This implies that x is invertible in A, contradicting the assumption that x is a zero divisor. Consequently, x is nilpotent. Thus the finite-dimensional K-algebra I is generated by a multiplicatively closed set of nilpotent elements and is hence nilpotent, see [R_1, 1.3.31] or [A, Corollary 1]. ∎

PROPOSITION 7.4: *Suppose a torus T acts algebraically on a central simple algebra $A = M_n(D)$. Then the fixed algebra A^T contains a complete collection of pairwise orthogonal primitive idempotents of A. That is, there exist primitive (in A) idempotents $e_1, \dots, e_n \in A^T$ such that $e_ie_j = \delta_{ij}e_i$ and $e_1 + e_2 + \cdots + e_n = 1$.*

Proof: We proceed by induction on the matrix size n of A. The assertion is trivially true if $n = 1$. If $n \geq 2$, Proposition 7.3 implies that A^T contains a nontrivial idempotent e. Set $f = 1 - e$. Then e and f are orthogonal idempotents. By [R_2, 1.1.12], the algebra eAe is simple. Note that its matrix size is strictly smaller than n. Since the action of T fixes e, T acts algebraically on eAe. By induction, $(eAe)^T$ contains a complete set of r pairwise orthogonal primitive idempotents $e_1, \dots, e_r$, where r is the Goldie rank of eAe. Similarly, $(fAf)^T$ contains a complete set of s pairwise orthogonal primitive idempotents $f_1, \dots, f_s$, where s is the Goldie rank of fAf. Thus the e_i and f_j form a set of pairwise orthogonal primitive idempotents of $A = M_n(D)$ such that their sum is 1. Consequently, $n = r + s$ (see [J, Chapter III, §10, Theorem 2, p.59]). ∎

Next we prove the main theorem of this section.

THEOREM 7.5: *Suppose a torus T acts algebraically on a central simple algebra A. Then one can write A in the form $A = M_n(D)$ where*

(a) *the standard matrix units are homogeneous for the T-action, and*

(b) *$D \cdot I_n$ is a division algebra which is stable under the action of T, and T acts algebraically on D.*

Here I_n is the $n \times n$-identity matrix.

Proof: By Proposition 7.4, A contains a complete collection of pairwise orthogonal idempotents $e_{11}, \ldots, e_{nn}$ which are fixed by T. We will first enlarge this set to a set of n^2 homogeneous matrix units e_{ij}.

Since A is simple, all $e_{ii}Ae_{jj}$ are non-zero. Also, $E = e_{11}Ae_{11}$ is a division algebra, and the $e_{jj}Ae_{11}$ (respectively, the $e_{11}Ae_{jj}$) are one-dimensional right (respectively, left) modules over E (see, e.g., [R_2, 2.1.21]). For $j = 2, \ldots, n$, choose a non-zero homogeneous element e_{1j} in $e_{11}Ae_{jj}$. Since $e_{jj}Ae_{11}$ is one-dimensional over E, it contains a unique element e_{j1} such that $e_{1j}e_{j1} = e_{11}$. The uniqueness of e_{j1} implies that e_{j1} is homogeneous. Note that $e_{j1}e_{11} = e_{j1}$, and $e_{11}e_{1j} = e_{1j}$. Since $e_{j1}e_{1j}$ is a non-zero idempotent in $e_{jj}Ae_{jj}$, we have $e_{j1}e_{1j} = e_{jj}$. We can thus consistently define $e_{ij} = e_{i1}e_{1j}$ for all $i, j = 1, \ldots, n$. Note that the e_{ij} are homogeneous, and that they are matrix units: $e_{1j}e_{k1} = \delta_{jk}e_{11}$ implies that $e_{ij}e_{kl} = \delta_{jk}e_{il}$.

Define a linear map $\rho\colon A \to A$ by $\rho(a) = \sum_u e_{u1}ae_{1u}$. One checks easily that the restriction of ρ to E respects multiplication. Thus $D = \rho(E) = \rho(A)$ is a division algebra isomorphic to E. Using the matrix units e_{ij}, one defines a k-algebra isomorphism $\varphi\colon A \to M_n(D)$ by sending $a \in A$ to the matrix (a_{ij}), where $a_{ij} = \sum_u e_{ui}ae_{ju}$ (see [R_2, 1.1.3]). Note that $\varphi(e_{ij})$ is the standard matrix unit with a 1 in the (i, j)-position and zeroes elsewhere. For $d \in D$, $\varphi(d) = d \cdot I_n$.

We now define the action of T on $M_n(D)$ by making φ equivariant. Then T acts algebraically on $M_n(D)$, and the standard matrix units are homogeneous (since they are the images of the e_{ij}). Since $e_{u1}e_{1u} = e_{11}$, and since e_{11} is fixed under the action of T, the characters of the homogeneous elements e_{u1} and e_{1u} are inverse to each other. This implies that ρ is equivariant. Thus D is T-stable, and so is $D \cdot I_n = \varphi(D)$.

In order to show that T acts algebraically on $D \cdot I_n$, it suffices to prove that it acts algebraically on $D = \rho(E)$. Since ρ is equivariant, we only need to check that the action of T on E is algebraic. For simplicity, write $e = e_{11}$ and thus $E = eAe$. Let $a \in E$. Then $a = (\sum a_i)/(\sum b_j)$, where the a_i and b_j are homogeneous elements of A and the b_j are central in A. Multiplying the equation $a(\sum b_j) = \sum a_i$ by e on both sides, and using the fact that $a = eae$, we obtain $a = (\sum ea_ie)/(\sum eb_je)$. Note that the ea_ie and eb_je are homogeneous elements contained in $E = eAe$ and the eb_je are central in E. Moreover, $\sum eb_je = e\sum b_j$ is non-zero since $\sum b_j$ is a non-zero central element. Hence, T acts algebraically on E (see Definition 5.2), completing the proof of the theorem. ∎

We now derive Theorem 1.1(b) from Theorem 7.5.

Proof of Theorem 1.1(b): A matrix algebra over a t-rational division algebra is clearly rational. Assume now that A is a t-rational central simple algebra. That is, there is a faithful algebraic action of a torus T of dimension $d + n - 1$ on A, where d is the transcendence degree of A over k and n is the matrix size of A. Write $A = M_n(D)$ as in Theorem 7.5. Denote by S the subgroup of T acting trivially on D. The connected component S° of S is again a torus, and $\dim T = \dim S^\circ + \dim T/S$. By Proposition 5.5, $\dim S^\circ \leq n-1$ and $\dim T/S \leq d$. Consequently, $\dim T/S = d$, and D is t-rational. ∎

8. Further properties of t-rational algebras

In this section we explore the consequences of Theorem 1.1. In particular, we prove Theorem 1.2. Recall that Theorem 2.4 applies to all t-rational division algebras.

COROLLARY 8.1: *Let D_1 and D_2 be division algebras with centers K_1 and K_2, respectively. Let $D = (D_1 \otimes_{K_1} K) \otimes_K (D_2 \otimes_{K_2} K)$, where K denotes the field compositum of K_1 and K_2.*

(a) *If D_1 and D_2 are t-rational division algebras then so is D.*

(b) *If D_1 and D_2 are stably t-rational then so is D.*

Proof: (a) Write D_1 and D_2 as basic products of symbols as in Definition 2.1(b) and combine the two decompositions. (b) Adjoin central indeterminates to D_1 and D_2 until they become t-rational; then apply part (a). ∎

COROLLARY 8.2: *Let n be a square-free integer.*

(a) *A division algebra D of degree n is cyclic if and only if it contains a t-rational division subalgebra E of degree n.*
(b) *The universal division algebra $D_{m,n}$ is a crossed product if and only if it contains a t-rational division subalgebra of degree n.*

Proof: (a) Assume that $E \subset D$ is t-rational. Then E is cyclic by Theorems 1.1 and 2.4(e). Hence so is D.

Conversely, suppose D is cyclic with center K. Then we can find non-zero elements x_1 and $x_2 \in D$ such that $a_1 = x_1^n \in K$, $a_2 = x_2^n \in K$, and $x_1x_2 = \omega x_2 x_1$ where ω is a primitive n-th root of unity. Tsen's theorem implies that a_1 and a_2 are algebraically independent; see [Pi, 19.4]. It is easy to see that the division subalgebra E of D generated by x_1 and x_2 (over k) is t-rational.

(b) Assume n is square-free. By a theorem of Amitsur the universal division algebra $D_{m,n}$ is a crossed product if and only if it is cyclic; see [R_1, 3.3.12]. Thus (b) follows from (a). ∎

Proof of Theorem 1.2(a): Assume that $D_{m,n}$ is a stably t-rational division algebra. Combining Theorems 1.1(a) and 2.4(a), we see that $\mathrm{char}(k) \nmid n$. By Theorem 3.1, $D_{m+1,n}$ is obtained from $D_{m,n}$ by adjoining n^2 central indeterminates. Thus $D_{m',n}$ is t-rational for a sufficiently large integer m'. We may therefore assume without loss of generality that $D_{m,n}$ is t-rational. By Theorem 2.4(d), the exponent of $D_{m,n}$ equals m_1. Since the exponent of $D_{m,n}$ is known to be n ([R_1, 3.2.8]) and $n = m_1 \cdots m_r$, we conclude that $r = 1$. Hence, by Theorem 2.4(b), $D_{m,n}$ is cyclic. By a theorem of Amitsur, this is impossible unless n is a product of distinct primes; see [R_1, 3.3.12]. ∎

Remark 8.3: A similar argument proves the following stronger assertion: If the universal division algebra $D_{m,n}$ contains a stably t-rational division subalgebra of degree n then n is a product of distinct primes.

Proof of Theorem 1.2(b): We use induction on the number r of primes dividing n. If $r = 1$, there is nothing to prove. Otherwise write $n = pa$ where p is a prime and a is a product of $r-1$ distinct primes. Theorem 3.2 says that $D_{m,n}$ is stably isomorphic to $D_{m,a} \otimes_K D_{m,p}$ where K is the compositum of $K_{m,a}$ and $K_{m,p}$ over k. We are assuming that $D_{m,p}$ is stably t-rational; by the induction assumption $D_{m,a}$ is stably t-rational as well. The conclusion of the theorem now follows from Corollary 8.1(b). ∎

While on the subject of primary components, we note the following corollary of Theorem 7.5.

COROLLARY 8.4: *The primary components of a t-rational division algebra are again t-rational.*

Proof: Let D be a t-rational division algebra via the faithful algebraic action of a torus T. Denote the center of D by K, and let E be a primary component of D. Then up to K-isomorphism, E is the underlying division algebra of some tensor power of D over K. That is, $D^{\otimes m} = M_n(E)$ for some positive integers m and n; see [R_1, 3.1.40]. The algebraic action of T on D extends diagonally to an algebraic action on $D^{\otimes m}$. Thus Theorem 7.5 allows us to assume that E is T-stable, and that T acts algebraically on E. Moreover, the action of T on the center K of E agrees with the original action of T on K. So by Lemma 5.10(b), the subgroup S of T acting trivially on E is finite. Thus T/S is a torus of the same dimension as T. Since E and D have the same transcendence degree over k, the action of T/S makes E into a t-rational division algebra. ∎

Remark 8.5: Our argument also proves the following. Let D_1 and D_2 be finite-dimensional division algebras with a common center K and let

$$A = D_1 \otimes_K D_2 = M_n(E),$$

where E is a division algebra. Assume that D_1 and D_2 are t-rational via faithful algebraic torus actions of the same torus T. If these actions agree on K, then both A and E are t-rational.

We conclude this section with another corollary of Theorem 7.5 which resembles a theorem of Bergman and Isaacs [BI] for actions of finite groups.

COROLLARY 8.6: *Let T be a torus acting algebraically on a central simple algebra A. Then A^T is semisimple Artinian.*

Proof: By Lemma 7.2, the nilradical N of A^T is nilpotent, and A^T/N is semisimple Artinian. It only remains to show that $N = 0$. Assume N contains a non-zero element x. Write $A = M_n(D)$ with homogeneous standard matrix units e_{ij} as in Theorem 7.5. Then for some integers i and j, $e_{ii}xe_{jj}$ is non-zero and belongs to N. Write $e_{ii}xe_{jj} = de_{ij}$ for some unique $0 \neq d \in D$. Denote by χ the character of e_{ij}. Since de_{ij} and $e_{jj} = e_{ji}e_{ij}$ belong to A^T, both d and e_{ji} are homogeneous with character χ^{-1}. Thus $d^{-1}e_{ji}$ is fixed, i.e., belongs to A^T. Hence the non-zero idempotent $e_{ii} = (de_{ij})(d^{-1}e_{ji})$ belongs to the nilpotent ideal N, a contradiction. ∎

9. Unirationality

Recall that throughout this paper $D_{m,n}$ denotes the universal division algebra of m-tuples of $n \times n$-matrices over k and $K_{m,n}$ denotes the center of $D_{m,n}$. In this section we construct a finite field extension F of $K_{2,n}$ of degree $(n-1)!$ such that $D_{2,n} \otimes F$ is a t-rational division algebra. We use this construction to prove that $D_{m,n}$ is t-unirational for any $m, n \geq 2$ (see Definition 9.3 and Corollary 9.5) and that $D_{m,2}$ is t-rational for any $m \geq 2$. We also give an explicit description of $D_{m,2}$ as the division algebra of fractions of a skew polynomial ring P^d_Ω; see Corollary 9.6.

Construction of F: Let $p(t)$ be the characteristic polynomial of the first generic matrix X_1 in $D_{2,n}$. Let $\alpha_1, \ldots, \alpha_n$ be the roots of this polynomial in the algebraic closure of $K = K_{2,n}$ and let $L = K(\alpha_1, \ldots, \alpha_n)$ be the splitting field. Then the Galois group of L over K is S_n. Let G be the cyclic subgroup of S_n generated by the n-cycle $(1\ 2\ \cdots\ n)$, and let $F = L^G$. Note that L is a cyclic extension of F of degree n.

THEOREM 9.1: *Assume* $\operatorname{char}(k) \nmid n$. *Then* $E = D_{2,n} \otimes_{K_{2,n}} F$ *is a t-rational division algebra.*

Proof: Since no non-trivial element of $G = <(1\ 2\ \cdots\ n)>$ fixes α_1, it follows that $\mathrm{Gal}(L/F(\alpha_1)) = \{1\}$. Hence $F(\alpha_1) = L$. Let h: $K(X_1) \longrightarrow L$ be the K-algebra homomorphism which sends X_1 to α_1. Counting dimensions, we see that $h \otimes \mathrm{id}$: $K(X_1) \otimes_K F \longrightarrow L$ is an isomorphism. In other words, $F(X_1)$ is isomorphic to L. In particular, $F(X_1)$ is a field.

Let $\beta_1 = X_1, \beta_2, \ldots, \beta_n$ be the elements of $F(X_1)$ corresponding to $\alpha_1, \ldots, \alpha_n \in L$. Let $\sigma \in \mathrm{Gal}(F(X_1)/F)$ be given by $\sigma(\beta_i) = \beta_{i+1}$ (mod n). Note that the group $\mathrm{Gal}(F(X_1)/F) =< \sigma >$ is cyclic of order n. For any $x \in F(X_1)$ we have

$$\mathrm{tr}(x) = x + \sigma(x) + \cdots + \sigma^{n-1}(x) \in F. \tag{4}$$

Note that the trace in E coincides with the trace in its maximal Galois subfield $F(X_1)$.

Recall that an element r of a ring R if called n-**central** if r^n lies in the center of R but r^i does not for any $i = 1, \ldots, n-1$.

LEMMA 9.2: *Let ω be a primitive n-th root of unity and let*

$$z = \beta_1 + \omega\beta_2 + \cdots + \omega^{n-1}\beta_n \in F(X_1).$$

Then

(a) $\mathrm{tr}(z^i) = 0$ *for* $i \not\equiv 0 \bmod n$,

(b) z *is n-central in* E,

(c) $F(X_1) = K(z)$, *and*

(d) $\mathrm{tr}(X_1 z^{-1}) = 1$.

Proof: (a) Note that $\sigma(z) = \omega^{-1}z$. Thus $\mathrm{tr}(z^i) = \mathrm{tr}(\sigma(z^i)) = \omega^{-i}\mathrm{tr}(z^i)$, implying that $\mathrm{tr}(z^i) = 0$ if $\omega^i \neq 1$.

(b) Since $\sigma(z^n) = z^n$, $z^n \in F$. Smaller powers of z cannot belong to F since their traces are zero by (a).

(c) Recall that $F(X_1) = K(\beta_1, \ldots, \beta_n)$, that the group $S_n = \mathrm{Gal}(F(X_1)/K)$ acts on $F(X_1)$ by permuting $\beta_1, \ldots, \beta_n$, and that K is the fixed field for that action. Thus we only need to show that $[K(z) : K] = n!$, i.e., that z has $n!$ distinct conjugates under the action of the symmetric group. This follows from the definition of z, since $1, \omega, \ldots, \omega^{n-1}$ are distinct, and since the β_i, being eigenvalues of the generic matrix X_1, are algebraically independent over k.

(d) Before we proceed with the proof, we remark that z is invertible, since it is a non-zero element of the field $F(X_1)$. Now apply formula (4) with $x = X_1 z^{-1} = \beta_1 z^{-1}$. Since $\sigma(z) = \omega^{-1}z$, we have $\sigma(z^{-1}) = \omega z^{-1}$ and thus

$$\mathrm{tr}(X_1 z^{-1}) = \beta_1 z^{-1} + \omega\beta_2 z^{-1} + \cdots + \omega^{n-1}\beta_n z^{-1} = zz^{-1} = 1. \qquad \blacksquare$$

Now let z be as in Lemma 9.2. Part (c) implies, in particular, that $F(z) = F(X_1)$. In other words, $1, z, \ldots, z^{n-1}$ generate $F(X_1)$ as an F-vector space, and thus we can write

$$X_1 = a_0 + a_1 z + \cdots + a_{n-1}z^{n-1} \tag{5}$$

where $a_0, \ldots, a_{n-1} \in F$. Since $\mathrm{tr}(z^i) = 0$ for $i \not\equiv 0 \bmod n$, we conclude that $a_1 = \frac{1}{n}\mathrm{tr}(X_1 z^{-1}) = \frac{1}{n}$ by Lemma 9.2(d).

By Lemma 9.2(b), z is n-central. Thus we can find an n-central invertible element $\tilde{w} \in E$ such that $z\tilde{w} = \omega\tilde{w}z$ for some primitive n-th root of unity ω; see [Pi, 15.1a]. Moreover, we can write the second generic matrix X_2 as

$$X_2 = y_0 + y_1 + \cdots + y_{n-1} \tag{6}$$

where $y_j \in \tilde{w}^j K(z)$. Write y_1 as $\tilde{w}r$ where r is an element of $F(z)$. If $y_1 \neq 0$ then $r \neq 0$ and, hence r and y_1 are invertible.

Now let $w = y_1$ if $y_1 \neq 0$ and $w = \tilde{w}$ otherwise. Then $zw = \omega wz$. Hence, w is non-singular and n-central. Since $zy_j = \omega^j y_j z$, we can write

$$y_j = \sum_{i=0}^{n-1} b_{ij} z^i w^j \tag{7}$$

for some $b_{ij} \in F$. Here $j = 0, 2, 3, \ldots, n-1$.

To summarize, we have chosen non-singular n-central elements $z, w \in E$ such that $zw = \omega wz$ and equations (5) — (7) hold with $a_1 = 1/n$ and $y_1 = 0$ or w.

Now let $P = P_\Omega^d = P_\Omega^{n^2+1}$ be as in Definition 2.2 with $\Omega = (\omega)$ (here $r = 1$). We will now show that E is isomorphic to the division algebra of fractions $Q_\Omega^{n^2+1}$ of P and therefore is t-rational; see Example 5.8. Indeed, we can write

$$P_\Omega^d = k[A_l, B_{ij}]\{Z, W\}/(ZW = \omega WZ),$$

where $i = 0, \ldots, n-1$ and $l, j = 0, 2, 3, \ldots, n-1$. Let φ: $P_\Omega^d \longrightarrow E$ be the k-algebra homomorphism given by $A_l \longrightarrow a_l$, $B_{ij} \longrightarrow b_{ij}$, $Z \longrightarrow z$ and $W \longrightarrow w$. Denote the image of this homomorphism by R. Let $Q(R)$ be the central simple subalgebra of E which is obtained from R by inverting all non-zero central elements. In order to finish the proof of the theorem it is sufficient to show that: (i) $Q(R) = E$ and (ii) φ is an isomorphism between P and R.

To prove (i) note that $Q(R)$ contains the generic matrices X_1 and X_2; see equations (5) - (7). Hence, $D_{2,n} \subset Q(R)$. Moreover, since $z \in R$, we also have $K(z) \subset Q(R)$. Recall that $K(z) = F(X_1)$; see Lemma 9.2(c). Since $Q(R)$ contains both $D_{2,n}$ and F, it has to be equal to all of E.

To prove (ii), denote the centers of P and R by $Z(P)$ and $Z(R)$ respectively. Since R and E have the same PI-degree, we have $\varphi(Z(P)) \subset Z(R)$. Since P is a finite $Z(P)$-module, R is a finite $\varphi(Z(P))$-module. Hence, by part (i)

$$\text{trdeg}_k \varphi(Z(P)) = \text{trdeg}_k Z(R) = \text{trdeg}_k F = n^2 + 1 = \text{trdeg}_k Z(P).$$

This shows that $\text{Ker}(\varphi) \cap Z(P) = (0)$. Hence, φ is injective; see [R_1, 1.6.27]. Since $\varphi(P) = R$, this completes the proof of (ii). ∎

Definition 9.3: A division algebra D of degree n is called **t-unirational** if it is contained in a t-rational division algebra of degree n.

One could also define a notion of t-unirationality by requiring D to be contained in a t-rational central simple algebra of the same degree. We chose the stronger condition 9.3 because it is more interesting in the case of universal division algebras. However, before we proceed, we remark that the two definitions are, in fact, distinct.

Example 9.4: A division subalgebra of a t-rational central simple algebra need not be t-unirational in the sense of Definition 9.3.

Let V be a d-dimensional vector space over k. Consider a faithful action of a finite non-abelian group G on V. Denote the field of rational functions of V by $L = k(V)$ and the field of invariants by $K = L^G$. By [FSS, 5.5], we may assume, after adding some indeterminates to L and K, that there exists a division algebra D with center K which contains L as a maximal subfield. Set $A = D \otimes_K L \approx M_n(L)$. Then A is t-rational, being matrices over a rational field. But the subalgebra D is not t-rational, since it is a crossed product for G and G is non-abelian; see Theorem 2.4(c). Moreover, if E is any division algebra containing D and of the same degree, then E is also a crossed product for G. Consequently, E cannot be t-rational either. ∎

COROLLARY 9.5: *Suppose that* $\text{char}\, k \nmid n$. *Then the universal division algebra* $D_{m,n}$ *is t-unirational for every* $n, m \geq 2$.

Proof: By Theorem 3.1 and Lemma 5.9, it is enough to show that $D_{2,n}$ is t-unirational. This is immediate from Theorem 9.1. ∎

We note that one can obtain a direct proof of Corollary 9.5 (i.e., one that does not involve an explicit construction of the extension $K_{m,n} \subset F$) by appealing to [RV_2, 3.1] or [Sa, Theorem 4].

Our final result is another corollary of Theorem 9.1.

COROLLARY 9.6: *Assume* $\text{char}\, k \neq 2$. *Then the universal division algebra* $D_{m,2}$ *is t-rational for any* $m \geq 2$. *In fact, set*

$$z = 2X_1 - \text{tr}(X_1) \quad \text{and} \quad w = \frac{1}{2}(X_2 - zX_2z^{-1}).$$

Then $D_{m,2}$ is the division algebra of fractions of

$$R = k\Big[\mathrm{tr}(X_1), \mathrm{tr}(X_2), \mathrm{tr}(X_1X_2), \mathrm{tr}(X_1^iX_2^jX_h)\Big]\{z, w\},$$

where $i, j = 0, 1$ and $h = 3, \ldots, m$. The ring R is isomorphic to the skew polynomial ring P_Ω^{4m-3} with $\Omega = (-1)$; see Definition 2.2.

Proof: When $n = 2$, the field F in Theorem 9.1 is equal to $K_{2,2}$. Thus $D_{2,2}$ is t-rational. Moreover, using the notation of the proof of Theorem 9.1 we see that $D_{2,2}$ is the division algebra of fractions of the skew polynomial ring $k[a_0, b_{00}, b_{10}]\{z, w\}$, where $a_0 = \frac{1}{2}\mathrm{tr}(X_1)$ and $z = 2X_1 - \mathrm{tr}(X_1)$. Since $X_2 = y_0 + w$, it follows that $zX_2z^{-1} = y_0 - w$ and thus $w = \frac{1}{2}(X_2 - zX_2z^{-1})$. Finally, $X_2 = y_0 + w = b_{00} + b_{10}z + w$. So $b_{00} = \frac{1}{2}\mathrm{tr}(X_2)$, and $2b_{10}z^2 = \mathrm{tr}(X_2z) = 2\mathrm{tr}(X_1X_2) - \mathrm{tr}(X_1)\mathrm{tr}(X_2)$. This proves the corollary for $m = 2$. The general case now follows from Theorem 3.1, since $D_{m,2} = D_{2,2}(\mathrm{tr}(X_1^iX_2^jX_h))$ as $i, j = 0, 1$ and $h = 3, \ldots, m$. ∎

We do not know whether or not $D_{m,n}$ is rational for square-free integers $n \geq 3$. Theorem 1.2(b) partially reduces this question to the case where n is a prime.

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